A NEW APPROACH TO THE FOURIER ANALYSIS ON SEMI-DIRECT PRODUCTS OF GROUPS

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ABSTRACT. Let H and K be locally compact groups and also $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$ be the semi-direct product of H and K with respect to the continuous homomorphism τ. This paper presents a novel approach to the Fourier analysis of G_τ, when K is abelian. We define the τ-dual group $G_{\widehat{\tau}}$ of G_τ as the semi-direct product $H \ltimes_{\widehat{\tau}} \widehat{K}$, where $\widehat{\tau} : H \to Aut(\widehat{K})$ defined via (3.1). We prove a Ponterjagin duality Theorem and also we study τ-Fourier transforms on G_τ. As a concrete application we show that how these techniques apply for the affine group and also we compute the τ-dual group of Euclidean groups and the Weyl-Heisenberg groups.

1. **Introduction**

Theory of Fourier analysis is the basic and fundamental step to extend the approximation theory on algebraic structures. Classical Fourier analysis on $\mathbb{R}^n$ and also it's standard extension for locally compact abelian groups play an important role in approximation theory and also time-frequency analysis. For more on this topics we refer the readers to [3] or [4]. Passing through the harmonic analysis of abelian groups to the harmonic analysis of non-abelian groups we loose many concepts of Fourier analysis on locally compact abelian groups. If we assume that G is unimodular and type I locally compact group, then still Fourier analysis on G can be used. Theory of Fourier analysis on non-abelian, unimodular and type I groups was completely studied by Lipsman in [9] and also Dixmier in [2] or [8].

Although theory of standard non-abelian Fourier analysis is a strong theory but it is not numerical computable, so it is not an appropriate tools in the view points of time-frequency analysis or physics and engineering applications. This lake persists us to have a new approach to the theory of Fourier analysis on non-abelian groups.

Many non-abelian groups which play important roles in general theory of time-frequency analysis or mathematical physics such as the affine group or Heisenberg group can be considered as a semi-direct products of some locally compact groups H and K with respect to a continuous homomorphism $\tau : H \to Aut(K)$ in which K is abelain.

In this paper which contains 5 sections, section 2 devoted to fix notations and also a summary of harmonic analysis on locally compact groups and semi-direct product of locally compact groups H and K with respect to the continuous homomorphism $\tau : H \to Aut(K)$. In section 3 we assume that K is abelian and also we define the τ-dual group $G_{\widehat{\tau}}$ of $G_\tau = H \ltimes_\tau K$ as the semi direct products of H and $\widehat{K}$ with respect to the continuous homomorphism $\widehat{\tau} : H \to Aut(\widehat{K})$, where $\widehat{\tau}_h(\omega) := \omega \circ \tau_{h^{-1}}$. It is also shown that the $\widehat{\tau}$-dual group $G_{\widehat{\widehat{\tau}}}$ of $G_{\widehat{\tau}} = H \ltimes_{\widehat{\tau}} K$ and G_τ are isomorphic, which can be considered as a generalization of the Ponterjagin duality Theorem.

In the sequel, in section 4 we define τ-Fourier transform of $f \in L^1(G_\tau)$ and we study it's basic L^2-properties such as the Plancherel theorem. We also prove an inversion formula for the τ-Fourier transform.

As well as, finally in section 5 as examples we show that how this extension techniques can be used for various types of semi-direct products of group such as the affine group, the Euclidean groups and the Weyl-Heisenberg groups.

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2. **Preliminaries and notations**

Let H and K be locally compact groups with identity elements e_H and e_K respectively and left Haar measures dh and dk respectively and also let $\tau : H \to Aut(K)$ be a homomorphism such that the map $(h,k) \mapsto \tau_h(k)$ from $H \times K$ onto K be continuous, where $Aut(K)$ is the group of all topological group automorphisms of K onto K. There is a natural topology, sometimes called Braconnier topology, turning $Aut(K)$ into a Hausdorff topological group(not necessarily locally compact), which is defined by the sub-base of identity neighbourhoods

$$\mathcal{B}(F,U) = \{\alpha \in Aut(K) : \alpha(k), \alpha^{-1}(k) \in Uk \ \forall k \in F\}, \tag{2.1}$$

where $F \subseteq K$ is compact and $U \subseteq K$ is an identity neighbourhood and also continuity of a homomorphism $\tau : H \to Aut(K)$ is equivalent to the continuity of the map $(h,k) \mapsto \tau_h(k)$ from $H \times K$ onto K (see [7]). The semi-direct product $G_\tau = H \ltimes_\tau K$ is a locally compact topological group with underlying set $H \times K$ which equipped with product topology and group operation is defined by

$$(h,k) \ltimes_\tau (h',k') := (hh', k\tau_h(k')) \quad \text{and} \quad (h,k)^{-1} := (h^{-1}, \tau_{h^{-1}}(k^{-1})). \tag{2.2}$$

If $H_1 := \{(h,e_K) : h \in H\}$ and $K_1 := \{(e_H,k) : k \in K\}$, then K_1 is a closed normal subgroup and H_1 is a closed subgroup of G_τ. The left Haar measure of G_τ is $d\mu_{G_\tau}(h,k) = \delta(h)dhdk$ and also $\Delta_{G_\tau}(h,k) = \delta(h)\Delta_H(h)\Delta_K(k)$, where the positive and continuous homomorphism $\delta : H \to (0,\infty)$ is given by (Theorem 15.29 of [5])

$$dk = \delta(h)d(\tau_h(k)). \tag{2.3}$$

From now on, for all $p \geq 1$ we denote by $L^p(G_\tau)$ the Banach space $L^p(G_\tau, \mu_{G_\tau})$ and also $L^p(K)$ stands for $L^p(K,dk)$. When $f \in L^p(G_\tau)$, for a.e. $h \in H$ the function f_h defined on K via $f_h(k) := f(h,k)$ belongs to $L^p(K)$ (see [4]).

If K is a locally compact abelian group, due to Corollary 3.6 of [3] all irreducible representations of K are one-dimensional. Thus, if π be an irreducible unitary representation of K we have $\mathcal{H}_\pi = \mathbb{C}$ and also according to the Shur's Lemma, there exists a continuous homomorphism ω of K into the circle group $\mathbb{T}$ such that for each $k \in K$ and $z \in \mathbb{C}$ we have $\pi(k)(z) = \omega(k)z$. Such continuous homomorphisms are called characters of K and the set of all characters of K denoted by $\widehat{K}$. If $\widehat{K}$ equipped by the topology of compact convergence on K which coincides with the w^*-topology that $\widehat{K}$ inherits as a subset of $L^\infty(K)$, then $\widehat{K}$ with respect to the dot product of characters is a locally compact abelian group which is called the dual group of K. The linear map $\mathcal{F}_K : L^1(K) \to \mathcal{C}(\widehat{K})$ defined by $v \mapsto \mathcal{F}_K(v)$ via

$$\mathcal{F}_K(v)(\omega) = \widehat{v}(\omega) = \int_K v(k)\overline{\omega(k)}dk, \tag{2.4}$$

is called the Fourier transform on K. It is a norm-decreasing $*$-homomorphism from $L^1(K)$ to $\mathcal{C}_0(\widehat{K})$ with a uniformly dense range in $\mathcal{C}_0(\widehat{K})$ (Proposition 4.13 of [3]). If $\phi \in L^1(\widehat{K})$, the function defined *a.e.* on K by

$$\check{\phi}(x) = \int_{\widehat{K}} \phi(\omega)\omega(x)d\omega, \tag{2.5}$$

belongs to $L^\infty(K)$ and also for all $f \in L^1(K)$ we have the following orthogonality relation (Parseval formula);

$$\int_K f(k)\overline{\check{\phi}(k)}dk = \int_{\widehat{K}} \widehat{f}(\omega)\overline{\phi(\omega)}d\omega. \tag{2.6}$$

The Fourier transform (2.4) on $L^1(K) \cap L^2(K)$ is an isometric and it extends uniquely to a unitary isomorphism from $L^2(K)$ to $L^2(\widehat{K})$ (Theorem 4.25 of [3]) and also each $v \in L^1(K)$ with $\widehat{v} \in L^1(\widehat{K})$ satisfies the following Fourier inversion formula (Theorem 4.32 of [3]);

$$v(k) = \int_{\widehat{K}} \widehat{v}(\omega)\omega(k)d\omega \text{ for a.e. } k \in K. \tag{2.7}$$

3. τ-Dual group

We recall that for a locally compact non-abelian group G, the standard dual space $\widehat{G}$ is defined as the set of all unitary equivalence classes of all irreducible unitary representations of G. There is a topology on $\widehat{G}$ called Fell topology. But $\widehat{G}$ with respect to the Fell topology is not a locally compact group in general setting (see [3]). On the other hand elements of $\widehat{G}$ are equivalence classes of irreducible unitary representations of G and so from computational view points there are not numerical applicable. In this section we associate to any semi-direct product group $H \ltimes_\tau K$ with K abelian, a τ-dual structure (group) which is actually a locally compact group.

Let H be a locally compact group and K be a locally compact abelian group also let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$. For all $h \in H$ and $\omega \in \widehat{K}$ define the action $H \times \widehat{K} \to \widehat{K}$ via

$$\omega_h := \omega \circ \tau_{h^{-1}}, \tag{3.1}$$

where $\omega_h(k) = \omega(\tau_{h^{-1}}(k))$ for all $k \in K$. If $\omega \in \widehat{K}$ and $h \in H$ we have $\omega_h \in \widehat{K}$, because for all $k, s \in K$ we have

$$\begin{aligned}
\omega_h(ks) &= \omega \circ \tau_{h^{-1}}(ks) \\
&= \omega(\tau_{h^{-1}}(ks)) \\
&= \omega(\tau_{h^{-1}}(k)\tau_{h^{-1}}(s)) \\
&= \omega(\tau_{h^{-1}}(k))\omega(\tau_{h^{-1}}(s)) = \omega_h(k)\omega_h(s).
\end{aligned}$$

In the following proposition we find a suitable relation about the Plancherel measure of $\widehat{K}$ and also the action of H on $\widehat{K}$ due to (3.1).

Proposition 3.1. *Let K be an abelian group and $\tau : H \to Aut(K)$ be a continuous homomorphism. The Plancherel measure $d\omega$ on $\widehat{K}$ for all $h \in H$ satisfies*

$$d\omega_h = \delta(h) d\omega, \tag{3.2}$$

where $\delta : H \to (0, \infty)$ is the positive continuous homomorphism given by $dk = \delta(h) d\tau_h(k)$.

Proof. Let $h \in H$ and also $v \in L^1(K)$. Using (2.3) we have $v \circ \tau_h \in L^1(K)$ with $\|v \circ \tau_h\|_{L^1(K)} = \delta(h)\|v\|_{L^1(K)}$, because

$$\begin{aligned}
\|v \circ \tau_h\|_{L^1(K)} &= \int_K |v \circ \tau_h(k)| dk \\
&= \int_K |v(\tau_h(k))| dk \\
&= \int_K |v(k)| d\tau_{h^{-1}}(k) \\
&= \delta(h) \int_K |v(k)| dk = \delta(h)\|v\|_{L^1(K)}.
\end{aligned}$$

Thus, for all $\omega \in \widehat{K}$ we achieve

$$\begin{aligned}
\widehat{v \circ \tau_h}(\omega) &= \int_K v(\tau_h(k))\overline{\omega(k)} dk \\
&= \int_K v(k)\overline{\omega_h(k)} d(\tau_{h^{-1}}(k)) \\
&= \delta(h) \int_K v(k)\overline{\omega_h(k)} dk = \delta(h)\widehat{v}(\omega_h).
\end{aligned}$$

Now let $v \in L^1(K) \cap L^2(K)$. According to the Plancherel theorem (Theorem 4.25 of [3]) and also preceding calculation, for all $h \in H$ we get

$$\begin{aligned}\int_{\widehat{K}} |\widehat{v}(\omega)|^2 d\omega_h &= \int_{\widehat{K}} |\widehat{v}(\omega_{h^{-1}})|^2 d\omega \\ &= \delta(h)^2 \int_{\widehat{K}} |\widehat{v \circ \tau_{h^{-1}}}(\omega)|^2 d\omega \\ &= \delta(h)^2 \int_K |v \circ \tau_{h^{-1}}(k)|^2 dk \\ &= \delta(h)^2 \int_K |v(k)|^2 d(\tau_h(k)) \\ &= \delta(h) \int_K |v(k)|^2 dk = \int_{\widehat{K}} |\widehat{v}(\omega)|^2 \delta(h) d\omega,\end{aligned}$$

which implies (3.2). □

Now using the action defined in (3.1) we define $\widehat{\tau} : H \to Aut(\widehat{K})$ via $h \mapsto \widehat{\tau}_h$, where

$$\widehat{\tau}_h(\omega) := \omega_h = \omega \circ \tau_{h^{-1}}. \tag{3.3}$$

According to (3.3) for all $h \in H$ we have $\widehat{\tau}_h \in Aut(\widehat{K})$. Because, if $k \in K$ and $h \in H$ then for all $\omega, \eta \in \widehat{K}$ we have

$$\begin{aligned}\widehat{\tau}_h(\omega.\eta)(k) &= (\omega.\eta)_h(k) \\ &= (\omega.\eta) \circ \tau_{h^{-1}}(k) \\ &= \omega.\eta(\tau_{h^{-1}}(k)) \\ &= \omega(\tau_{h^{-1}}(k))\eta(\tau_{h^{-1}}(k)) \\ &= \omega_h(k)\eta_h(k) = \widehat{\tau}_h(\omega)(k)\widehat{\tau}_h(\eta)(k).\end{aligned}$$

Also $h \mapsto \widehat{\tau}_h$ is a homomorphism from H into $Aut(\widehat{K})$, cause if $h, t \in H$ then for all $\omega \in \widehat{K}$ and also $k \in K$ we have

$$\begin{aligned}\widehat{\tau}_{th}(\omega)(k) &= \omega_{th}(k) \\ &= \omega(\tau_{(th)^{-1}}(k)) \\ &= \omega(\tau_{h^{-1}}\tau_{t^{-1}}(k)) \\ &= \omega_h(\tau_{t^{-1}}(k)) \\ &= \widehat{\tau}_h(\omega)(\tau_{t^{-1}}(k)) = \widehat{\tau}_t[\widehat{\tau}_h(\omega)](k).\end{aligned}$$

Thus, via an algebraic viewpoint we can consider the semi-direct product of H and $\widehat{K}$ with respect to the homomorphism $\widehat{\tau} : H \to Aut(\widehat{K})$. Due to (2.2), $\widehat{\tau}$-dual group operation for all $(h, \omega), (t, \eta) \in G_{\widehat{\tau}} = H \ltimes_{\widehat{\tau}} \widehat{K}$ is

$$(h, \omega) \ltimes_{\widehat{\tau}} (t, \eta) = (ht, \omega.\eta_h). \tag{3.4}$$

Now we are in the position to prove the following fundamental theorem.

Theorem 3.2. *Let H and K be locally compact groups with K abelian, $\tau : H \to Aut(K)$ be a continuous homomorphism and also let $\delta : H \to (0, \infty)$ be the positive continuous homomorphism given via $dk = \delta(h)d\tau_h(k)$. The homomorphism $\widehat{\tau} : H \to Aut(\widehat{K})$ defined in (3.1) is continuous and so that the semi-direct product $H \ltimes_{\widehat{\tau}} \widehat{K}$ is a locally compact group with the left Haar measure $d\mu_{G_{\widehat{\tau}}}(h, \omega) = \delta(h)^{-1} dh d\omega$.*

Proof. For $\alpha \in Aut(K)$ let $\widehat{\alpha} \in Aut(\widehat{K})$ be given for all $\omega \in \widehat{K}$ by $\widehat{\alpha}(\omega) := \omega \circ \alpha^{-1}$ where for all $k \in K$ we have $\omega \circ \alpha^{-1}(k) = \omega(\alpha^{-1}(k))$. Due to Theorem 26.9 and also Theorem 26.5 of [5] the mapping $\widehat{\ } : Aut(K) \to Aut(\widehat{K})$ defined by $\alpha \mapsto \widehat{\alpha}$ is a topological group isomorphism and so it is continuous. According to the following diagram

$$H \xrightarrow{\tau} Aut(K) \xrightarrow{\widehat{\ }} Aut(\widehat{K}), \tag{3.5}$$

the homomorphism $\widehat{\tau} : H \to Aut(\widehat{K})$ defined in (3.1) is continuous. Thus, the semi-direct product $H \ltimes_{\widehat{\tau}} \widehat{K}$ is a locally compact group and also Proposition 3.1 shows that $\delta(h)^{-1}dhd\omega$ is a left Haar measure for $H \ltimes_{\widehat{\tau}} \widehat{K}$. □

The semi-direct product $G_{\widehat{\tau}} = H \ltimes_{\widehat{\tau}} \widehat{K}$ mentioned in Theorem 3.2, called the *τ-dual group* of $G_\tau = H \ltimes_\tau K$. The most important advantage of this definition as a kind of a dual space for semi direct product of locally compact groups is that its elements are numerical computable and also this dual space is merely a locally compact group. It is worthwhile to note that, when H is the identity group, the τ-dual group of $G_\tau = K$ coincides with the usual dual group $\widehat{K}$ of K. When K is abelian locally compact group and $\tau : H \to Aut(K)$ is a continuous homomorphism, we call K as the Fourier factor of the semi-direct product $G_\tau = H \ltimes_\tau K$.

Due to the Pontrjagin duality theorem (Theorem 4.31 of [3]), each $k \in K$ defines a character $\widehat{k}$ on $\widehat{K}$ via $\widehat{k}(\omega) = \omega(k)$ and also the map $k \mapsto \widehat{k}$ is a topological group isomorphism from K onto $\widehat{\widehat{K}}$. Via the same method as introduced in (3.1) the $\widehat{\tau}$-dual group operation, for all $(h, \widehat{k})$ and $(t, \widehat{s})$ in $G_{\widehat{\widehat{\tau}}} = H \ltimes_{\widehat{\widehat{\tau}}} \widehat{\widehat{K}}$ is

$$(h, \widehat{k}) \ltimes_{\widehat{\widehat{\tau}}} (t, \widehat{s}) = (ht, \widehat{k}\widehat{\widehat{\tau}}_h(\widehat{s})), \tag{3.6}$$

where $\widehat{\widehat{\tau}} : H \to Aut(\widehat{\widehat{K}})$ is given by

$$\widehat{\widehat{\tau}}_h(\widehat{k})(\omega) = \omega_{h^{-1}}(k), \tag{3.7}$$

for all $\omega \in \widehat{K}$ and $(h, k) \in G_\tau$. Because, due to (3.3) we have

$$\begin{aligned}
\widehat{\widehat{\tau}}_h(\widehat{k})(\omega) &= \widehat{k} \circ \widehat{\tau}_{h^{-1}}(\omega) \\
&= \widehat{k}\left(\widehat{\tau}_{h^{-1}}(\omega)\right) \\
&= \widehat{k}(\omega_{h^{-1}}) = \omega_{h^{-1}}(k).
\end{aligned}$$

In the sequel we prove a type of Pontrjagin duality theorem for τ-dual group of semi direct product of groups. But first we prove a short lemma.

Lemma 3.3. *Let K be an abelian group and $\tau : H \to Aut(K)$ be a continuous homomorphism also let $G_\tau = H \ltimes_\tau K$. Then, for all $(h, k) \in G_\tau$ we have*

$$\widehat{\tau_h(k)} = \widehat{\widehat{\tau}}_h(\widehat{k}). \tag{3.8}$$

Proof. Let $(h, k) \in G_\tau$ and also let $\omega \in \widehat{K}$. Using duality notation and also (3.7) we have

$$\begin{aligned}
\widehat{\tau_h(k)}(\omega) &= \omega(\tau_h(k)) \\
&= \omega \circ \tau_h(k) \\
&= \omega_{h^{-1}}(k) = \widehat{\widehat{\tau}}_h(\widehat{k})(\omega).
\end{aligned}$$

□

Next theorem gives us a subtle topological group isomorphism form G_τ onto $G_{\widehat{\widehat{\tau}}}$. In fact, the next theorem can be considered as the Pontrjagin duality theorem for τ-dual group of semi-direct product of groups.

Theorem 3.4. *Let H be a locally compact group, K be a locally compact abelian group and $\tau : H \to Aut(K)$ be a continuous homomorphism also let $G_\tau = H \ltimes_\tau K$. The map $\Theta : G_\tau \to G_{\widehat{\widehat{\tau}}}$ defined by*

$$(h, k) \mapsto \Theta(h, k) := (h, \widehat{k}), \tag{3.9}$$

is a topological group isomorphism.

Proof. First we show that Θ is a homomorphism. Let $(h,k),(t,s)$ in G_τ. Since the map $k\mapsto\widehat{k}$ is a homomorphism and also using Lemma 3.3 we have

$$\begin{aligned}\Theta\left((h,k)\ltimes_\tau(t,s)\right)&=\Theta\left(ht,k\tau_h(s)\right)\\&=\left(ht,\widehat{k\tau_h(s)}\right)\\&=\left(ht,\widehat{k}\widehat{\tau_h(s)}\right)\\&=\left(ht,\widehat{k}\widehat{\widehat{\tau}}_h(\widehat{s})\right)\\&=(h,\widehat{k})\ltimes_{\widehat{\widehat{\tau}}}(t,\widehat{s})=\Theta(h,k)\ltimes_{\widehat{\widehat{\tau}}}\Theta(t,s).\end{aligned}$$

Now using Pontrjagin Theorem (Theorem 4.31 of [3]), the map $k\mapsto\widehat{k}$ is a topological group isomorphism from K onto $\widehat{\widehat{K}}$ which implies that the map Θ is also a homeomorphism. Thus Θ is a topological group isomorphism. □

Remark 3.5. From now on due to Theorem 3.4 we can identify $G_{\widehat{\widehat{\tau}}}$ with G_τ via the topological group isomorphism Θ defined in (3.9). More precisely, we may identify an element $(h,\widehat{k})\in G_{\widehat{\widehat{\tau}}}$ with (h,k).

4. τ-Fourier transform

In this section we study the τ-Fourier analysis on the semi-direct product G_τ.

We define the τ-*Fourier transform* of $f\in L^1(G_\tau)$ for a.e. $(h,\omega)\in G_{\widehat{\tau}}$ by

$$\mathcal{F}_\tau(f)(h,\omega):=\delta(h)\mathcal{F}_K(f_h)(\omega)=\delta(h)\int_K f(h,k)\overline{\omega(k)}dk. \tag{4.1}$$

In the next theorem we prove a Parseval formula for the τ-Fourier transform.

Theorem 4.1. *Let $\tau:H\to Aut(K)$ be a continuous homomorphism and $G_\tau=H\ltimes_\tau K$ with K abelian also let $f\in L^1(G_\tau)$ and $\Psi\in L^1(G_{\widehat{\tau}})$. Define the function g for a.e. $(h,k)\in G_\tau$ by*

$$g(h,k):=\int_{\widehat{K}}\Psi(h,\omega)\overline{\omega(k)}d\omega. \tag{4.2}$$

Then, $\mathcal{F}_\tau(f)_h$ belongs to $L^\infty(\widehat{K})$ and g_h belongs to $L^\infty(K)$ for a.e. $h\in H$ also we have the following orthogonality relations;

$$\int_{G_\tau}\delta(h)^{-1}f(h,k)\overline{g(h,k)}d\mu_{G_\tau}(h,k)=\int_{G_{\widehat{\tau}}}\mathcal{F}_\tau(f)(h,\omega)\overline{\Psi(h,\omega)}d\mu_{G_{\widehat{\tau}}}(h,\omega), \tag{4.3}$$

$$\int_{G_\tau}f(h,k)\overline{g(h,k)}d\mu_{G_\tau}(h,k)=\int_{G_{\widehat{\tau}}}\delta(h)\mathcal{F}_\tau(f)(h,\omega)\overline{\Psi(h,\omega)}d\mu_{G_{\widehat{\tau}}}(h,\omega). \tag{4.4}$$

Proof. Let $f\in L^1(G_\tau)$ and $\Psi\in L^1(G_{\widehat{\tau}})$. It is clear that for a.e. $h\in H$ we have $\mathcal{F}_\tau(f)_h\in L^\infty(\widehat{K})$ and $g_h\in L^\infty(K)$. Using Parseval Theorem (2.6), we get

$$\int_K f(h,k)\overline{g(h,k)}dk=\int_{\widehat{K}}\mathcal{F}_K(f_h)(\omega)\overline{\Psi(h,\omega)}d\omega. \tag{4.5}$$

Thus by (4.5) we have

$$\begin{aligned}\int_{G_\tau}\delta(h)^{-1}f(h,k)\overline{g(h,k)}d\mu_{G_\tau}(h,k)&=\int_H\left(\int_K f(h,k)\overline{g(h,k)}dk\right)dh\\&=\int_H\left(\int_{\widehat{K}}\mathcal{F}_K(f_h)(\omega)\overline{\Psi(h,\omega)}d\omega\right)dh=\int_{G_{\widehat{\tau}}}\mathcal{F}_\tau(f)(h,\omega)\overline{\Psi(h,\omega)}d\mu_{G_{\widehat{\tau}}}(h,\omega).\end{aligned}$$

The same argument and also (4.5) implies (4.4). □

Due to (4.1), if $f \in L^2(G_\tau)$ we have $f_h \in L^2(K)$ for a.e. $h \in H$. Thus, according to Theorem 4.25 of [3], $\mathcal{F}_K(f_h)$ is well-defined for a.e. $h \in H$. Now, in the following theorem we show that the τ-Fourier transform (4.1) is a unitary transform from $L^2(G_\tau)$ onto $L^2(G_{\widehat{\tau}})$.

The next theorem can be considered as a Plancherel formula for the τ-Fourier transform.

Theorem 4.2. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$ with K abelian. The τ-Fourier transform (4.1) on $L^2(G_\tau)$ is an isometric transform from $L^2(G_\tau)$ onto $L^2(G_{\widehat{\tau}})$.*

Proof. Let $f \in L^2(G_\tau)$. Using Fubini's theorem and also Plancherel theorem (Theorem 4.25 of [3]) we have

$$\begin{aligned}
\|\mathcal{F}_\tau(f)\|^2_{L^2(G_{\widehat{\tau}})} &= \int_{G_{\widehat{\tau}}} |\mathcal{F}_\tau(f)(h,\omega)|^2 d\mu_{G_{\widehat{\tau}}}(h,\omega) \\
&= \int_H \left(\int_{\widehat{K}} |\mathcal{F}_K(f_h)(\omega)|^2 d\omega \right) \delta(h) dh \\
&= \int_H \left(\int_K |f_h(k)|^2 dk \right) \delta(h) dh \\
&= \int_H \int_K |f(h,k)|^2 \delta(h) dk dh \\
&= \int_{G_\tau} |f(h,k)|^2 d\mu_{G_\tau}(h,k) = \|f\|^2_{L^2(G_\tau)}.
\end{aligned}$$

Therefore, the linear map $f \mapsto \mathcal{F}_\tau(f)$ is an isometric in the L^2-norm. Now we show that, it is also surjective. Let $\phi \in L^2(G_{\widehat{\tau}})$. Then, for a.e. $h \in H$ we have $\phi_h \in L^2(\widehat{K})$. Again using the Plancherel Theorem, there is a unique $v^h \in L^2(K)$ such that we have $\mathcal{F}_K(v^h) = \phi_h$. Put $f(h,k) := \delta(h)^{-1} v^h(k)$, then we have $f \in L^2(G_\tau)$. Because,

$$\begin{aligned}
\int_{G_\tau} |f(h,k)|^2 d\mu_{G_\tau}(h,k) &= \int_H \int_K |f(h,k)|^2 \delta(h) dh dk \\
&= \int_H \left(\int_K |v^h(k)|^2 dk \right) \delta(h)^{-1} dh \\
&= \int_H \left(\int_{\widehat{K}} |\mathcal{F}_K(v^h)(\omega)|^2 d\omega \right) \delta(h)^{-1} dh \\
&= \int_H \left(\int_{\widehat{K}} |\phi_h(\omega)|^2 d\omega \right) \delta(h) dh \\
&= \int_{G_{\widehat{\tau}}} |\phi(h,\omega)|^2 d\mu_{G_{\widehat{\tau}}}(h,\omega) < \infty.
\end{aligned}$$

Also, for a.e. $(h,\omega) \in G_{\widehat{\tau}}$ we have

$$\begin{aligned}
\mathcal{F}_\tau(f)(h,\omega) &= \delta(h)\mathcal{F}_K(f_h)(\omega) \\
&= \mathcal{F}_K(v^h)(\omega) \\
&= \phi_h(\omega) = \phi(h,\omega).
\end{aligned}$$

□

Now we can prove the following Fourier inversion theorem for the τ-Fourier transform defined in (4.1).

Theorem 4.3. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$ with K abelian also let $f \in L^1(G_\tau)$ with $\mathcal{F}_\tau(f) \in L^1(G_{\widehat{\tau}})$. Then, for a.e. $(h,k) \in G_\tau$ we have the following reconstruction formula;*

$$f(h,k) = \delta(h)^{-1} \int_{\widehat{K}} \mathcal{F}_\tau(f)(h,\omega)\omega(k) d\omega. \tag{4.6}$$

Proof. Let $f \in L^1(G_\tau)$ with $\mathcal{F}_\tau(f) \in L^1(G_{\widehat{\tau}})$. Then, for a.e. $h \in H$ we have $f_h \in L^1(K)$ and $\mathcal{F}_K(f_h) \in L^1(\widehat{K})$. Using Theorem 4.32 of [3], for a.e. $(h,k) \in G_\tau$ we have

$$\begin{aligned} f(h,k) &= \int_{\widehat{K}} \mathcal{F}_K(f_h)(\omega)\omega(k)d\omega \\ &= \delta(h)^{-1}\int_{\widehat{K}} \delta(h)\mathcal{F}_K(f_h)(\omega)\omega(k)d\omega = \delta(h)^{-1}\int_{\widehat{K}} \mathcal{F}_\tau(f)(h,\omega)\omega(k)d\omega. \end{aligned}$$

□

We can also define the *generalized τ-Fourier transform* of $f \in L^1(G_\tau)$ for a.e. $(h,\omega) \in G_{\widehat{\tau}}$ by

$$\mathcal{F}_\tau^\sharp(f)(h,\omega) := \delta(h)^{3/2}\mathcal{F}_K(f_h)(\omega_h) = \delta(h)^{3/2}\int_K f(h,k)\overline{\omega_h(k)}dk. \tag{4.7}$$

The following Parseval formula for the generalized τ-Fourier transform can be also proved.

Theorem 4.4. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$ with K abelian also let $f \in L^1(G_\tau)$ and $\Psi \in L^1(G_{\widehat{\tau}})$. Define the function g for a.e. $(h,k) \in G_\tau$ by*

$$g(h,k) := \int_{\widehat{K}} \Psi(h,\omega)\omega_h(k)d\omega. \tag{4.8}$$

Then, $\mathcal{F}_\tau^\sharp(f)_h$ belongs to $L^\infty(\widehat{K})$ and g_h belongs to $L^\infty(K)$ for a.e. $h \in H$ also we have the following orthogonality relations;

$$\int_{G_\tau} \delta(h)^{-1/2} f(h,k)\overline{g(h,k)}d\mu_{G_\tau}(h,k) = \int_{G_{\widehat{\tau}}} \mathcal{F}_\tau^\sharp(f)(h,\omega)\overline{\Psi(h,\omega)}d\mu_{G_{\widehat{\tau}}}(h,\omega), \tag{4.9}$$

$$\int_{G_\tau} f(h,k)\overline{g(h,k)}d\mu_{G_\tau}(h,k) = \int_{G_{\widehat{\tau}}} \delta(h)^{1/2}\mathcal{F}_\tau^\sharp(f)(h,\omega)\overline{\Psi(h,\omega)}d\mu_{G_{\widehat{\tau}}}(h,\omega). \tag{4.10}$$

Proof. It is easy to check that for a.e. $h \in H$, $\mathcal{F}_\tau^\sharp(f)_h$ belongs to $L^\infty(\widehat{K})$ and also g_h belongs to $L^\infty(K)$. Using Fubini's Theorem and also the standard Parseval formula (2.6) for a.e. $h \in H$ we get

$$\begin{aligned} \int_K f(h,k)\overline{g(h,k)}dk &= \int_K f(h,k)\left(\int_{\widehat{K}} \overline{\Psi(h,\omega)}\overline{\omega_h}(k)d\omega\right)dk \\ &= \int_{\widehat{K}}\left(\int_K f_h(k)\overline{\omega_h}(k)dk\right)\overline{\Psi(h,\omega)}d\omega \\ &= \int_{\widehat{K}} \widehat{f_h}(\omega_h)\overline{\Psi(h,\omega)}d\omega = \delta(h)^{-3/2}\int_{\widehat{K}} \mathcal{F}_\tau^\sharp(f)(h,\omega)\overline{\Psi(h,\omega)}d\omega. \end{aligned}$$

Now, we achieve

$$\begin{aligned} \int_{G_\tau} \delta(h)^{-1/2} f(h,k)\overline{g(h,k)}d\mu_{G_\tau}(h,k) &= \int_H \left(\int_K f(h,k)\overline{g(h,k)}dk\right)\delta(h)^{1/2}dh \\ &= \int_H \left(\int_{\widehat{K}} \mathcal{F}_\tau^\sharp(f)(h,\omega)\overline{\Psi(h,\omega)}d\omega\right)\delta(h)^{-1}dh = \int_{G_{\widehat{\tau}}} \mathcal{F}_\tau^\sharp(f)(h,\omega)\overline{\Psi(h,\omega)}d\mu_{G_{\widehat{\tau}}}(h,\omega). \end{aligned}$$

The same method implies (4.10). □

If we choose f in $L^2(G_\tau)$, then for a.e. $h \in H$ we have $f_h \in L^2(K)$ and so that according to Theorem 4.25 of [3], $\mathcal{F}_K(f_h)$ belongs to $L^2(\widehat{K})$. In the following theorem, we show that the generalized τ-Fourier transform (4.7) is a unitary transform from $L^2(G_\tau)$ onto $L^2(G_{\widehat{\tau}})$.

Theorem 4.5. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$ with K abelian. The generalized τ-Fourier transform (4.7) is an isometric transform from $L^2(G_\tau)$ onto $L^2(G_{\widehat{\tau}})$.*

Proof. Let $f \in L^2(G_\tau)$. Due to Proposition 3.1, Fubini's theorem and also Plancherel theorem (Theorem 4.25 of [3]) we have

$$\begin{aligned}
\|\mathcal{F}_\tau^\sharp(f)\|^2_{L^2(G_{\widehat{\tau}})} &= \int_{G_{\widehat{\tau}}} |\mathcal{F}_\tau^\sharp(f)(h,\omega)|^2 d\mu_{G_{\widehat{\tau}}}(h,k) \\
&= \int_H \left(\int_{\widehat{K}} |\mathcal{F}_\tau^\sharp(f)(h,\omega)|^2 dk \right) \delta(h)^{-1} dh \\
&= \int_H \left(\int_{\widehat{K}} |\mathcal{F}_K(f_h)(\omega_h)|^2 d\omega \right) \delta(h)^2 dh \\
&= \int_H \left(\int_{\widehat{K}} |\mathcal{F}_K(f_h)(\omega)|^2 d\omega_{h^{-1}} \right) \delta(h)^2 dh \\
&= \int_H \left(\int_{\widehat{K}} |\mathcal{F}_K(f_h)(\omega)|^2 d\omega \right) \delta(h) dh \\
&= \int_H \left(\int_K |f(h,k)|^2 dk \right) \delta(h) dh \\
&= \int_{G_\tau} |f(h,k)|^2 d\mu_{G_\tau}(h,k) = \|f\|^2_{L^2(G_\tau)}.
\end{aligned}$$

Now to show that the generalized τ-Fourier transform (4.7) maps $L^2(G_\tau)$ onto $L^2(G_{\widehat{\tau}})$, let $\phi \in L^2(G_{\widehat{\tau}})$ be given. Then, for a.e. $h \in H$ we have $\phi_h \in L^2(K)$ and so that there is unique $v^h \in L^2(K)$ with $\mathcal{F}_K(v^h) = \phi_h$. Put $f(h,k) = \delta(h)^{-1/2} v^h \circ \tau_{h^{-1}}(k)$. Then we have $f \in L^2(G_\tau)$, because

$$\begin{aligned}
\|f\|^2_{L^2(G_\tau)} &= \int_{G_\tau} |f(h,k)|^2 d\mu_{G_\tau}(h,k) \\
&= \int_H \left(\int_K |f(h,k)|^2 dk \right) \delta(h) dh \\
&= \int_H \left(\int_K |v^h \circ \tau_{h^{-1}}(k)|^2 dk \right) dh \\
&= \int_H \left(\int_{\widehat{K}} |\mathcal{F}_K(v^h \circ \tau_{h^{-1}})(\omega)|^2 d\omega \right) dh \\
&= \int_H \left(\int_{\widehat{K}} |\mathcal{F}_K(v^h)(\omega_{h^{-1}})|^2 d\omega \right) \delta(h)^{-2} dh \\
&= \int_H \left(\int_{\widehat{K}} |\mathcal{F}_K(v^h)(\omega)|^2 d\omega_h \right) \delta(h)^{-2} dh \\
&= \int_H \left(\int_{\widehat{K}} |\mathcal{F}_K(v^h)(\omega)|^2 d\omega \right) \delta(h)^{-1} dh \\
&= \int_H \left(\int_{\widehat{K}} |\phi(h,\omega)|^2 d\omega \right) \delta(h)^{-1} dh = \|\phi\|^2_{L^2(G_{\widehat{\tau}})}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\mathcal{F}_\tau^\sharp(f) &= \delta(h)^{3/2} \mathcal{F}_K(f_h)(\omega_h) \\
&= \delta(h) \mathcal{F}_K(v^h \circ \tau_{h^{-1}})(\omega_h) \\
&= \mathcal{F}_K(v^h)(\omega) = \phi(h,\omega).
\end{aligned}$$

□

In the following we prove an inversion formula for the generalized τ-Fourier transform defined in (4.7).

Theorem 4.6. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$ with K abelian also let $f \in L^1(G_\tau)$ with $\mathcal{F}^\sharp_\tau(f) \in L^1(G_{\widehat{\tau}})$. Then, for a.e. $(h,k) \in G_\tau$ we have the following reconstruction formula;*

$$f(h,k) = \delta(h)^{-1/2} \int_{\widehat{K}} \mathcal{F}^\sharp_\tau(f)(h,\omega)\omega_h(k)d\omega. \tag{4.11}$$

Proof. Let $f \in L^1(G_\tau)$ and also let $\mathcal{F}^\sharp_\tau(f) \in L^1(G_{\widehat{\tau}})$. Due to Proposition 3.1, for a.e. $h \in H$ we have

$$\int_{\widehat{K}} |\mathcal{F}_K(f_h)(\omega_h)|d\omega = \int_{\widehat{K}} |\mathcal{F}_K(f_h)(\omega)|d\omega_{h^{-1}} = \delta(h)^{-1} \int_{\widehat{K}} |\mathcal{F}_K(f_h)(\omega)|d\omega.$$

Thus, for a.e. $h \in H$ we get $\mathcal{F}_K(f_h) \in L^1(\widehat{K})$. Now, using Theorem 4.32 of [3] and also Proposition 3.1 for a.e. $(h,k) \in G_\tau$ we achieve

$$\begin{aligned} f(h,k) &= \int_{\widehat{K}} \mathcal{F}_K(f_h)(\omega)\omega(k)d\omega \\ &= \int_{\widehat{K}} \mathcal{F}_K(f_h)(\omega_h)\omega_h(k)d\omega_h \\ &= \delta(h) \int_{\widehat{K}} \mathcal{F}_K(f_h)(\omega_h)\omega_h(k)d\omega \\ &= \delta(h)^{-1/2} \int_{\widehat{K}} \delta(h)^{3/2}\mathcal{F}_K(f_h)(\omega_h)\omega_h(k)d\omega = \delta(h)^{-1/2} \int_{\widehat{K}} \mathcal{F}^\sharp_\tau(f)(h,\omega)\omega_h(k)d\omega. \end{aligned}$$

□

5. **Examples**

As an application we study the theory of τ-Fourier transform for the affine group $\mathbf{a}x + \mathbf{b}$.

5.1. **Affine group $\mathbf{ax} + \mathbf{b}$.** Let $H = \mathbb{R}^*_+ = (0, +\infty)$ and $K = \mathbb{R}$. The affine group $a\mathbf{x} + b$ is the semi direct product $H \ltimes_\tau K$ with respect to the homomorphism $\tau : H \to Aut(K)$ given by $a \mapsto \tau_a$, where $\tau_a(b) = ab$. Hence the underlying manifold of the affine group is $(0,\infty) \times \mathbb{R}$ and also the group law is

$$(a,b) \ltimes_\tau (a',b') = (aa', b + ab'). \tag{5.1}$$

The continuous homomorphism $\delta : H \to (0,\infty)$ is given by $\delta(a) = a^{-1}$ and so that the left Haar measure is in fact $d\mu_{G_\tau}(a,b) = a^{-2}dadb$. Due to Theorem 4.5 of [3] we can identify $\widehat{\mathbb{R}}$ with $\mathbb{R}$ via $\omega(b) = \langle b, \omega\rangle = e^{2\pi i\omega b}$ for each $\omega \in \widehat{\mathbb{R}}$ and so we can consider the continuous homomorphism $\widehat{\tau} : H \to Aut(\widehat{K})$ given by $a \mapsto \widehat{\tau}_a$ via

$$\begin{aligned} \langle b, \widehat{\tau}_a(\omega)\rangle &= \langle b, \omega_a\rangle \\ &= \langle \tau_{a^{-1}}(b), \omega\rangle = \langle a^{-1}b, \omega\rangle = e^{2\pi i\omega a^{-1}b}. \end{aligned}$$

Thus, τ-dual group of the affine group again has the underlying manifold $(0,\infty) \times \mathbb{R}$, with τ-dual group law given by

$$(a,\omega) \ltimes_{\widehat{\tau}} (a',\omega') = (aa', \omega + \omega'_a) = (aa', \omega + a^{-1}\omega'). \tag{5.2}$$

Using Theorem 3.2, the left Haar measure $d\mu_{G_{\widehat{\tau}}}(a,\omega)$ of $G_{\widehat{\tau}}$ is precisely $dad\omega$. Now we recall that the standard dual space of the affine group which is precisely the set of all unitary irreducible representations of the affine group $a\mathbf{x} + b$, are described via Theorem 6.42 of [3] and also Theorem 7.50 of [3] guarantee the following Plancherel formula;

$$\|\widehat{f}(\pi_+)\|^2_{\mathrm{HS}} + \|\widehat{f}(\pi_-)\|^2_{\mathrm{HS}} = \int_0^\infty \int_{-\infty}^{+\infty} \frac{|f(a,b)|^2}{a^2} dadb, \tag{5.3}$$

for all measurable function $f : (0,\infty) \times \mathbb{R} \to \mathbb{C}$ satisfying

$$\int_0^\infty \int_{-\infty}^{+\infty} \frac{|f(a,b)|^2}{a^2} dadb < \infty. \tag{5.4}$$

Now let $f:(0,\infty)\times\mathbb{R}\to\mathbb{C}$ be a measurable function satisfying

$$\int_0^\infty\int_{-\infty}^{+\infty}\frac{|f(a,b)|}{a^2}dadb<\infty. \tag{5.5}$$

Then, for a.e. $(a,\omega)\in G_{\widehat{\tau}}=\mathbb{R}_+^*\ltimes_{\widehat{\tau}}\mathbb{R}$ the τ-Fourier transform of f is given by (4.1) via

$$\mathcal{F}_\tau(f)(a,\omega)=\delta(a)\widehat{f_a}(\omega)=a^{-1}\int_{-\infty}^{+\infty}f(a,b)e^{-2\pi i\omega.b}db,$$

and also the following Plancherel formula for the τ-Fourier transform holds;

$$\int_0^\infty\int_{-\infty}^{+\infty}|\mathcal{F}_\tau(f)(a,\omega)|^2dad\omega=\int_0^\infty\int_{-\infty}^{+\infty}\frac{|f(a,b)|^2}{a^2}dadb. \tag{5.6}$$

Therefore, we have the following proposition.

Proposition 5.1. *Let $f:(0,\infty)\times\mathbb{R}\to\mathbb{C}$ be a measurable function satisfying (5.4). Then,*

$$\int_0^\infty\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(a,b)\overline{f(a,\beta)}}{a^2}e^{-2\pi i\omega(b-\beta)}dbd\beta dad\omega=\int_0^\infty\int_{-\infty}^{+\infty}\frac{|f(a,b)|^2}{a^2}dadb. \tag{5.7}$$

Proof. Using (5.6) and also Fubini's theorem we have

$$\begin{aligned}\int_0^\infty\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(a,b)\overline{f(a,\beta)}}{a^2}e^{-2\pi i\omega(b-\beta)}dbd\beta dad\omega&=\int_0^\infty\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty}f(a,b)e^{-2\pi i\omega.b}db\right|^2\frac{dad\omega}{a^2}\\&=\int_0^\infty\int_{-\infty}^{+\infty}|\mathcal{F}_\tau(f)(a,\omega)|^2dad\omega=\int_0^\infty\int_{-\infty}^{+\infty}\frac{|f(a,b)|^2}{a^2}dadb.\end{aligned}$$

□

Due to the reconstruction formula (4.6) each measurable function $f:(0,\infty)\times\mathbb{R}\to\mathbb{C}$ satisfying (5.5) with

$$\int_0^\infty\int_{-\infty}^{+\infty}|\mathcal{F}_\tau(f)(a,\omega)|dad\omega<\infty, \tag{5.8}$$

satisfies the following reconstruction formula;

$$\begin{aligned}f(a,b)&=\delta(a)^{-1}\int_{-\infty}^{+\infty}\widehat{f_a}(\omega)e^{2\pi i\omega.b}d\omega\\&=\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}f(a,\beta)e^{-2\pi i\omega\beta}d\beta\right)e^{2\pi i\omega.b}d\omega=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(a,\beta)e^{2\pi i\omega(b-\beta)}d\beta d\omega.\end{aligned}$$

For a.e. $(a,\omega)\in G_{\widehat{\tau}}=\mathbb{R}_+^*\ltimes_{\widehat{\tau}}\mathbb{R}$ and also each measurable function $f:(0,\infty)\times\mathbb{R}\to\mathbb{C}$ satisfying (5.4), the generalized τ-Fourier transform of f is given by (4.7) via

$$\begin{aligned}\mathcal{F}_\tau^\sharp(f)(a,\omega)&=\delta(a)^{3/2}\widehat{f_a}(\omega_a)\\&=a^{-3/2}\int_{-\infty}^{+\infty}f(a,b)e^{-2\pi i\omega_a.b}db=a^{-3/2}\int_{-\infty}^{+\infty}f(a,b)e^{-2\pi i\omega a^{-1}b}db.\end{aligned}$$

According to Theorem 4.5 the generalized τ-Fourier transform satisfies the following Plancherel formula;

$$\int_0^\infty\int_{-\infty}^{+\infty}|\mathcal{F}_\tau^\sharp(f)(a,\omega)|^2dad\omega=\int_0^\infty\int_{-\infty}^{+\infty}\frac{|f(a,b)|^2}{a^2}dadb. \tag{5.9}$$

Proposition 5.2. *Let $f:(0,\infty)\times\mathbb{R}\to\mathbb{C}$ be a measurable function satisfying (5.4). Then,*

$$\int_0^\infty\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(a,b)\overline{f(a,\beta)}}{a^3}e^{-2\pi i\omega a^{-1}(b-\beta)}dbd\beta dad\omega=\int_0^\infty\int_{-\infty}^{+\infty}\frac{|f(a,b)|^2}{a^2}dadb. \tag{5.10}$$

Proof. Using (5.9) and also Fubini's theorem we have

$$\begin{aligned}\int_0^\infty\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{f(a,b)\overline{f(a,\beta)}}{a^3}e^{-2\pi i\omega a^{-1}(b-\beta)}dbd\beta dad\omega &= \int_0^\infty\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty}f(a,b)e^{-2\pi i\omega a^{-1}b}db\right|^2\frac{dad\omega}{a^3}\\ &= \int_0^\infty\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty}f(a,b)e^{-2\pi i\omega_a.b}db\right|^2\frac{dad\omega}{a^3}\\ &= \int_0^\infty\int_{-\infty}^{+\infty}|\mathcal{F}_\tau^\sharp(f)(a,\omega)|^2dad\omega = \int_0^\infty\int_{-\infty}^{+\infty}\frac{|f(a,b)|^2}{a^2}dadb.\end{aligned}$$

□

Due to the reconstruction formula (4.11) each measurable function $f:(0,\infty)\times\mathbb{R}\to\mathbb{C}$ satisfying (5.5) with

$$\int_0^\infty\int_{-\infty}^{+\infty}|\mathcal{F}_\tau^\sharp(f)(a,\omega)|dad\omega<\infty, \tag{5.11}$$

satisfies the following reconstruction formula;

$$\begin{aligned}f(a,b) &= \delta(a)^{-1/2}\int_{-\infty}^{+\infty}\mathcal{F}_\tau^\sharp(f)(a,\omega)\omega_a(b)d\omega\\ &= \delta(a)\int_{-\infty}^{+\infty}\widehat{f}_a(\omega_a)\omega_a(b)d\omega\\ &= \delta(a)\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}f(a,\beta)e^{-2\pi i\omega_a}d\beta\right)\omega_a(b)d\omega\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(a,\beta)e^{-2\pi i\omega_a.(\beta-b)}d\beta d\omega = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(a,\beta)e^{-2\pi i\omega a^{-1}(\beta-b)}d\beta d\omega.\end{aligned}$$

In the sequel we find the τ-dual group of some other well-known semi-direct product groups.

5.2. **Euclidean groups.** Let $E(n)$ be the group of rigid motions of $\mathbb{R}^n$, the group generated by rotations and translations. If we put $H=\mathrm{SO}(n)$ and also $K=\mathbb{R}^n$, then $E(n)$ is the semi direct product of H and K with respect to the continuous homomorphism $\tau:\mathrm{SO}(n)\to Aut(\mathbb{R}^n)$ given by $\sigma\mapsto\tau_\sigma$ via $\tau_\sigma(\mathbf{x})=\sigma\mathbf{x}$ for all $\mathbf{x}\in\mathbb{R}^n$. The group operation for $E(n)$ is

$$(\sigma,\mathbf{x})\ltimes_\tau(\sigma',\mathbf{x}')=(\sigma\sigma',\mathbf{x}+\tau_\sigma(\mathbf{x}'))=(\sigma\sigma',\mathbf{x}+\sigma\mathbf{x}'). \tag{5.12}$$

Identifying $\widehat{K}$ with $\mathbb{R}$, the continuous homomorphism $\widehat{\tau}:\mathrm{SO}(n)\to Aut(\mathbb{R}^n)$ is given by $\sigma\mapsto\widehat{\tau}_\sigma$ via

$$\begin{aligned}\langle\mathbf{x},\mathbf{w}_\sigma\rangle &= \langle\mathbf{x},\widehat{\tau}_\sigma(\mathbf{w})\rangle\\ &= \langle\tau_{\sigma^{-1}}(\mathbf{x}),\mathbf{w}\rangle=\langle\sigma^{-1}\mathbf{x},\mathbf{w}\rangle=e^{-2\pi i(\sigma^{-1}\mathbf{x},\mathbf{w})}=e^{-2\pi i(\sigma\mathbf{x},\mathbf{w})}.\end{aligned}$$

where $(.,.)$ stands for the standard inner product of $\mathbb{R}^n$. Since H is compact we have $\delta=1$ and so that $d\sigma d\mathbf{x}$ is a left Haar measure for $E(n)$. Thus, the τ-dual group of $E(n)$ has underlying manifold $\mathrm{SO}(n)\times\mathbb{R}^n$ with the group operation

$$(\sigma,\mathbf{w})\ltimes_{\widehat{\tau}}(\sigma',\mathbf{w}')=(\sigma\sigma',\mathbf{w}+\mathbf{w}'_\sigma). \tag{5.13}$$

5.3. **The Weyl-Heisenberg group.** Let K be a locally compact abelian (LCA) group with the Haar measure dk and $\widehat{K}$ be the dual group of K with the Haar measure $d\omega$ also $\mathbb{T}$ be the circle group and let the continuous homomorphism $\tau:K\to Aut(\widehat{K}\times\mathbb{T})$ via $s\mapsto\tau_s$ be given by $\tau_s(\omega,z)=(\omega,z.\omega(s))$. The semi-direct product $G_\tau=K\ltimes_\tau(\widehat{K}\times\mathbb{T})$ is called the Weyl-Heisenberg group associated with K which is usually denoted by $\mathbb{H}(K)$. The group operation for all $(k,\omega,z),(k',\omega',z')\in K\ltimes_\tau(\widehat{K}\times\mathbb{T})$ is

$$(k,\omega,z)\ltimes_\tau(k',\omega',z')=(k+k',\omega\omega',zz'\omega'(k)). \tag{5.14}$$

If dz is the Haar measure of the circle group, then $dkd\omega dz$ is a Haar measure for the Weyl-Heisenberg group and also the continuous homomorphism $\delta : K \to (0,\infty)$ given in (2.3) is the constant function 1. Thus, using Theorem 4.5 and also Proposition 4.6 of [3] and also Theorem 3.2 we can obtain the continuous homomorphism $\widehat{\tau} : K \to Aut(K\times\mathbb{Z})$ via $s \mapsto \widehat{\tau}_s$, where $\widehat{\tau}_s$ is given by $\widehat{\tau}_s(k,n) = (k,n)\circ\tau_{s^{-1}}$ for all $(k,n)\in K\times\mathbb{Z}$ and $s\in K$. Due to Theorem 4.5 of [3], for each $(k,n)\in K\times\mathbb{Z}$ and also for all $(\omega,z)\in\widehat{K}\times\mathbb{T}$ we have

$$\begin{aligned}\langle(\omega,z),(k,n)_s\rangle &= \langle(\omega,z),\widehat{\tau}_s(k,n)\rangle\\ &= \langle\tau_{s^{-1}}(\omega,z),(k,n)\rangle\\ &= \langle(\omega,z\overline{\omega(s)}),(k,n)\rangle\\ &= \langle\omega,k\rangle\langle z\overline{\omega(s)},n\rangle\\ &= \omega(k)z^n\overline{\omega(s)}^n\\ &= \omega(k-ns)z^n = \langle\omega,k-ns\rangle\langle z,n\rangle = \langle(\omega,z),(k-ns,n)\rangle.\end{aligned}$$

Thus, $(k,n)_s = (k-ns,n)$ for all $k,s\in K$ and $n\in\mathbb{Z}$. Therefore, $G_{\widehat{\tau}}$ has the underlying set $K\times K\times\mathbb{Z}$ with the following group operation;

$$\begin{aligned}(s,k,n)\ltimes_{\widehat{\tau}}(s',k',n') &= (s+s',(k,n)\widehat{\tau}_s(k',n'))\\ &= (s+s',(k,n)(k'-n's,n')) = (s+s',k+k'-n's,n+n').\end{aligned}$$

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